A Simple Nonparametric Approach to the Term Structure of Credit Default Swap Spreads

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Abstract

This study introduces a nonparametric approach to pricing credit default swaps (CDSs). This method is notable for its simplicity, estimation speed, and flexibility. That is, it relies exclusively on closed-form solutions (which provide instantaneous results) and allows the user to reproduce any term structure of CDS spreads. I empirically assess its pricing performance by comparing it with an otherwise equivalent semiparametric (piecewise constant default probability) model that requires a series of root-search algorithms and represents the current market convention for marking-to-market CDS contracts. This analysis demonstrates that the new method also implies a reduction in mean percentage absolute pricing errors.

JEL classification: G12; G13.

Keywords: Credit risk pricing, no-arbitrage conditions, bootstrapping, CDS contracts.

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1. Introduction

The term structure of credit default swap (CDS) spreads represents valuable information for pricing credit-risky securities: mainly (and unsurprisingly) the actual positions in CDS contracts. Pricing models based on the term structure of CDS spreads can be classified as either parametric or semiparametric. Several studies apply parametric models (e.g., Chen et al. 2013; Jarrow et al. 2019; Pan and Singleton 2008), and they generally work as follows. First, the researcher assumes a stochastic (parametric) process for the risk-neutral default intensity and a distribution function for CDS spread pricing errors. Second, based on these assumptions, the model parameters are estimated using the maximum likelihood method or a similar optimization rule. Finally, the estimated model can be used to price existing CDS contracts and other credit-risky securities (e.g., risky bonds). An appealing characteristic of parametric models is that all prices rely on a few parameter values, which in turn can be modeled as a function of fundamental economic variables. For the same reason, the main limitation of these models is that pricing errors can be minimized, but never completely eliminated. That is, using these models for pricing necessarily assumes some degree of market mispricing in the observed CDS spreads, model mispricing, or a combination of the two.[1]

Semiparametric models are the conventional approach to marking-to-market CDS contracts. While different variations exist (Duffie 1999; Hull and White 2003; O'Kane and Turnbull 2003), these models share several core assumptions. The risk-free interest rate process and default time are risk-neutrally independent, and (forward risk-neutral) default probabilities have a piecewise constant profile. Based on these assumptions, the term structure of default probabilities can be estimated sequentially from the earliest to latest maturity of available CDS spreads such that the model perfectly refits these observed quotes. Because a constant default probability model represents the clearest example of a parametric

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[1] This discussion refers to so-called reduced-form models (Jarrow and Turnbull 1995). Structural credit risk models (Merton 1974) constitute a different family of parametric models. Du et al. (2019) offer a good example of the technical challenges associated with replicating the observed term structure of CDS spreads based on a structural credit risk model.

model, a piecewise constant default probability (PWCDP) model can be effectively described as semiparametric. However, this alternative approach also has costs. Operationally, this means solving a sequence of root-search algorithms equal to the number of observed (or otherwise specified) quotes. This process is time-consuming and does not guarantee convergence in all cases. In terms of accuracy, it is sensible to require a CDS pricing model to replicate the observed CDS spreads. However, a model's ability to price CDS contracts with different, mostly untraded, maturities is directly related to its precision in assessing the spreads that the market would eventually agree upon for those maturities. Therefore, we may question the accuracy of the semiparametric approach in estimating *unobserved* CDS spreads, whether there is room for improvement, and at what cost (if any).

Figure 1 illustrates an example. Figure 1A presents the hypothetical *complete* term structure of CDS spreads (CTSCDS; the black solid line, left axis). Generated using a particular parameterization of the Svensson model ($\beta_0 = 140$; $\beta_1 = -133$; $\beta_2 = -325$; $\beta_3 = 275$; $\alpha_1 = 2.2$; $\alpha_2 = 3.1$), this comprises all possible maturities over a 30-year horizon, from one to 10,950 calendar dates. However, the CTSCDS is not observed in practice. As the figure indicates, the *observed* term structure of CDS spreads (OTSCDS; red points, left axis) is typically reduced to 6m, 1y, 2y, 3y, 4y, 5y, 7y, 10y, 15y, 20y, and 30y.[2] Figure 1A incorporates the predicted CTSCDS according to the PWCDP model discussed later (blue dashed line, left axis) and corresponding percentage absolute pricing errors (PAPEs; black dotted line, right axis). The PWCDP model offers a perfect fit for the observed CDS spreads, although noticeable pricing errors may exist for the remainder of the curve. The example differentiates between lower (blue area) and higher (green area) maturities than the most liquid 5y term. This differentiation is relevant. According to Bank for International Settlements data (second half of 2022), single-name CDS contracts with a remaining maturity equal to or less than five years represent 92.70% of the total notional amount outstanding.

[2] I chose the parameter values in the Svensson model to closely fit the observed quotes for a representative company. Please refer to the median CDS spreads in Table 2.

Figure 1B presents the same *true* CTSCDS as in Figure 1A; however, I use the OTSCDS in this case to perform a straight interpolation between the observed quotes, specifically through the Shape-Preserving Piecewise Cubic Hermite Interpolation (PCHIP; i.e., the Matlab® code for this interpolation scheme). As the figure clarifies, the PCHIP method offers a more accurate representation of the true CTSCDS than the PWCDP model. The improvement applies also (and especially) for the most common contract maturities. As this example shows, even a relatively simple interpolation scheme may provide a better fit for the CTSCDS than the PWCDP model. By extension, the example also suggests that instead of assuming a piecewise constant default probability ex-ante and estimating the CTSCDS ex-post, the problem of model mispricing can be minimized by first fitting the most plausible CTSCDS based on the observed quotes and having a pricing model capable of reproducing the entire curve. Certainly, these competing strategies entail a potential tradeoff. If pricing errors from the semiparametric approach are economically insignificant (i.e., below the observed bid-ask spreads), a marginal gain in accuracy may be insufficient to justify a more complex or computationally demanding model. However, if a more accurate model is noticeably simpler and faster to implement, then the proposed model clearly outperforms the traditional semiparametric approach.

<Figure 1 about here>

This study contributes to the literature on credit risk pricing in general and that on the pricing of CDS contracts in particular by deriving an extremely simple, nonparametric pricing model that allows the user to refit any pre-specified CTSCDS. The model draws on three core elements. First, the price of a CDS contract can always be expressed as a simple function of a reduced number of well-established building blocks in credit risk pricing or credit risk discount factors (CRDFs), initially defined by Lando (1998). Second, I can show that in a discrete-time economy in which all future asset maturities and possible defaulting times are the same (i.e., all future calendar dates, consistent with the proposed interpolation), a set of no-arbitrage conditions must hold between the values of these CRDFs for any two consecutive maturities. Finally, based on these results, a system of equations exists that allows the immediate bootstrapping of such CRDFs for all possible maturities. Specifically, the bootstrapping procedure is based exclusively on closed-form solutions. Thus, unlike the semiparametric approach, it does not involve a sequence of root-search algorithms. Overall, the nonparametric model that I introduce in this study is more flexible, simpler, and faster than the conventional semiparametric approach and, therefore, more efficient.

To address pricing errors, I analyze a sample of 104 companies with highly liquid CDS spreads over 2010–2019 to compare the performance of four pricing approaches: the benchmark PWCDP model and three versions of the nonparametric model introduced here in which the CTSCDS is estimated ex-ante using a linear, PCHIP, or cubic spline interpolation (termed the NP/Linear, NP/PCHIP, and NP/Spline models hereafter). Based on this empirical analysis, I conclude that the nonparametric model with a PCHIP interpolation provides the lowest mean PAPE (MPAPE), while the PWCDP model generates the highest MPAPE. While a comparison with typical bid-ask spreads suggests that this gain in accuracy is not economically significant, it is an additional benefit of the new method. Further, it concentrates precisely on the maturities segment that, in practice, is most relevant for pricing (i.e., those equal to or less than the standard 5-year term).

The remainder of this paper proceeds as follows. Section 2 defines the basic setting and introduces the no-arbitrage conditions for the CRDFs. Section 3 reviews CDS contract pricing based on the CRDFs. Section 4 incorporates the additional assumptions and describes the bootstrapping process. Section 5 presents a conventional PWCDP model as a restricted case. Section 6 discusses the possible applications of the term structure of the CRDFs, primarily marking-to-market CDS contracts. Section 7 presents an empirical analysis that ranks the accuracy of the four pricing approaches. Finally, Section 8 summarizes the main conclusions.

2. Basic Setting and No-Arbitrage Conditions for the Credit Risk Discount Factors

2.1. Setting

I focus on the pricing of CDS contracts and other single-name credit-risky securities at the current (non-defaulting) time 0. The assumption is that of a simple discrete-time economy with daily time intervals. Traded assets include (but are not restricted to) default-free and risky zero-coupon bonds of all possible maturities.[3] I denote the maturities as T, which correspond to all future calendar dates up to time τ; that is, $T \in \{\Delta, 2\Delta, \dots, \tau\}$, with $\Delta = 1/365$. The price of a default-free zero-coupon bond with a nominal value of \$1 and maturity T is $Z(T)$.[4] For risky bonds, default may occur at any future calendar date and represents an absorbing state. The default time is denoted τ^d, while the minimum between τ^d and T is denoted L_d^T. In the case of default, bond holders receive (irrespective of the possible coupon) a fraction θ of its face value and the asset is liquidated.[5] The markets are complete and arbitrage-free.

2.2. Credit Risk Discount Factors and No-Arbitrage Conditions

In this setting, the three basic CRDFs are defined as follows:

- $A(T)$: The present value of asset class A paying a constant annuity of \$$\Delta$ every Δ years until L_d^T (included).
- $B(T)$: The present value of asset class B paying \$1 at τ^d, provided $\tau^d \leq T$.
- $C(T)$: The present value of asset class C paying \$1 at T, provided $\tau^d > T$.

I stress here that, for asset class A with maturity T, default at $\tau^d \leq T$ implies the cancelation of the periodic stream of payments from $\tau^d + \Delta$ onward. This includes $\tau^d + \Delta$, but not τ^d itself. Although this clarification is meaningless in a continuous-time model (Lando 1998), it is a key element in this case. In addition, my discrete-time setting enables the introduction of a fourth convenient CRDF:

- $E(T)$: The present value of asset class E paying \$1 at T, provided $\tau^d > T - \Delta$.

Hence, the difference between assets C and E with the same maturity T is that the payment of \$1 at T is conditional on survival at time T for the former asset and on survival at the previous date $T - \Delta$ for the latter.

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[3] The latter assumption is made for convenience and can be easily relaxed. In particular, as in Jarrow and Turnbull (1995), the only real requirement is that sufficient traded assets exist to allow the prices of default-free and risky zero-coupon bonds to recover for all possible maturities.

[4] As all prices are determined at time 0, I use simple notation to avoid re-emphasizing time 0. Additionally, $Z(T) \equiv e^{-r(T)T}$, where $r(T)$ is the spot rate with maturity T.

[5] The recovery of face value assumption is standard in CDS pricing models and consistent with actual practice.

Figure 2 depicts the payment structures of the four contingent claims. Along with the assumptions in Section 2.1, these payment structures imply two no-arbitrage conditions that must hold for any two consecutive maturities, $T - \Delta$ and T.

<Figure 2 about here>

The first no-arbitrage condition (NAC1) relates $A(T)$, $A(T - \Delta)$, and $E(T)$:

$$A(T) = A(T - \Delta) + \Delta E(T), \tag{1}$$

with $A(0) = 0$.

In Equation (1), the present value of a daily annuity of \$$\Delta$ paid until time T or default must be equal to the sum of (a) the present value of the daily annuity of \$$\Delta$ paid until time $T - \Delta$ or default and (b) the present value of \$$\Delta$ paid with certainty at time T, conditional on no default at time $T - \Delta$ or before. The second component follows from the previous comment on the effect of default on asset A payments.

The second no-arbitrage condition (NAC2), which must hold for any two consecutive maturities $T - \Delta$ and T, is

$$C(T) + B(T) - B(T - \Delta) = E(T), \tag{2}$$

with $B(0) = 0$.

On the left-hand side of Equation (2), $C(T)$ is the present value of \$1 paid at time T, conditional on no default at that time or before. In addition, $B(T) - B(T - \Delta)$ equals the present value of \$1 paid at time T in the case of default at that moment and not before. Taken as a whole, the left-hand side of Equation (2) equals the present value of \$1 paid with certainty at time T, conditional on no default at time $T - \Delta$ or before, exactly what $E(T)$ on the right-hand side of the said equation represents. Combining Equations (1) and (2) yields the following related condition:

$$A(T) = A(T - \Delta) + \Delta[C(T) + B(T) - B(T - \Delta)]. \tag{3}$$

Equation (3) provides the necessary relationship between the three core CRDFs for any two consecutive maturities $T - \Delta$ and T. Importantly, this equilibrium condition relies exclusively on the payment structures of assets A, B, and C and the assumptions in Section 2.1. That is, it does not depend on a risk-neutral pricing model. A further intuitive implication of Equation (3) is

$$A(T) = \Delta\left[\sum_{h=1}^{T/\Delta} C(h\Delta) + B(T)\right]. \tag{4}$$

3. Credit Default Swap Spreads as a Function of Credit Risk Discount Factors

The value of a position in a CDS contract with maturity T is the difference between its premium and protection legs. Figure 3 illustrates the daily structure of the premium leg and reflects a key feature of a CDS contract. While the annual premium per dollar of protected debt, cds, is generally paid in quarterly installments, the liquidation of the contract in the case of default implies the payment of the premium accrued since the last quarterly payment. Hence, a non-defaulting state on a given day implies a consolidated right to accrue Δcds the following day, regardless of whether default occurs on that day. If we further assume no counterparty risk from the protection buyer's side, then the consolidated right to accrue Δcds can be considered to be risk-free income on a given day, conditional on no default on the previous day. Because this payment structure mimics that of asset A, scaled by cds, the present value of the premium leg is simply

$$X(T) = cdsA(T), \tag{5}$$

where the nominal value of the protected bond is normalized to 1.

<Figure 3 about here>

Figure 4 depicts the daily structure of the protection leg. On any given day, the protection payment is 0 in the case of no default and a fraction $(1-\theta)$ of the protected bond's face value in the case of default. Thus, the payment structure of the protection leg reproduces that of asset B scaled by $(1-\theta)$, and the same applies for its present value for a nominal of 1:

$$Y(T) = (1-\theta)B(T). \tag{6}$$

<Figure 4 about here>

Finally, we obtain the break-even CDS spread, $cds(T)$, by equating the premium and protection legs of the contract (see also Duffie 1999):

$$cds(T) = \frac{(1-\theta)B(T)}{A(T)}. \tag{7}$$

4. Additional Assumptions and Bootstrapping the Credit Risk Discount Factors

All the previous results are based on no-arbitrage arguments alone, implying that they do not rely on risk-neutral pricing models. However, a convenient additional assumption is that the risk-free interest rate process and default time are risk-neutrally independent (Jarrow and Turnbull 1995; Jarrow et al. 1997; Duffie 1999; Hull and White 2003; O'Kane and Turnbull 2003). If we denote $S(T)$ as the risk-neutral survival probability at time T (as seen at current time 0), this new assumption allows us to decompose $C(T-\Delta)$ and $E(T)$ as follows: $C(T-\Delta) = Z(T-\Delta)S(T-\Delta)$; and $E(T) = Z(T)S(T-\Delta)$. If we further denote $f(T-\Delta, T) \equiv -(1/\Delta)log[Z(T)/Z(T-\Delta)]$ as the forward (risk-free) rate between $T-\Delta$ and T, we have

$$E(T) = e^{-f(T-\Delta,T)\Delta}C(T-\Delta), \tag{8}$$

with $C(0) = Z(0) = 1$. The interpretation of this equation is straightforward. Under the assumption of risk-neutral independence between the risk-free interest rate process and default time, we can obtain $E(T)$ by discounting first from T to $T-\Delta$ at the forward rate and then from $T-\Delta$ to 0 using the discount factor $C(T-\Delta)$.

Let us now assume that $A(T-\Delta)$, $B(T-\Delta)$, and $C(T-\Delta)$ are available for a given maturity $T-\Delta$. In this case, and assuming that the forward rate $f(T-\Delta, T)$ is also available, Equations (1), (2), (7), and (8) lead to a system of three equations and three unknowns—$A(T)$, $B(T)$, and $C(T)$—with a simple closed-form solution:

$$A(T) = A(T-\Delta) + \Delta e^{-f(T-\Delta,T)\Delta}C(T-\Delta); \tag{9a}$$

$$B(T) = \frac{cds(T)A(T)}{(1-\theta)}; \tag{9b}$$

$$C(T) = e^{-f(T-\Delta,T)\Delta}C(T-\Delta) - B(T) + B(T-\Delta). \tag{9c}$$

Several aspects of this finding require special attention. First, because Equation System (9) links $\{A(T), B(T), C(T)\}$ to $\{A(T-\Delta), B(T-\Delta), C(T-\Delta)\}$, we can bootstrap the full term structure of

CRDFs based on a previously settled CTSCDS and the initial values $\{A(0), B(0), C(0)\} = \{0,0,1\}$. Second, because the solution is also in closed form (trivial and unique), this estimation does not require a series of root-search algorithms or any other optimization rule. That is, we can obtain the full term structure of CRDFs instantaneously using a spreadsheet. Third, these term structures converge naturally toward their risk-free counterparts as the CTSCDS tends to a flat zero curve: $B(T)$ tends to zero, $C(T)$ tends to $Z(T)$, and $A(T)$ tends to $\Delta \sum_{h=1}^{T/\Delta} Z(h\Delta)$.[6] Fourth, the solution contains no specific assumptions about the risk-free interest rate process or default time. Their unique assumption is that they are risk-neutrally independent. Finally, the solution does not even involve estimating the risk-neutral survival (or forward default) probabilities. As I demonstrate in the next section, they can be easily obtained as a *sub-product* of the bootstrapping process. However, these additional results are not required for any application considered below.

Table 1 provides a numerical example in which I estimate the CTSCDS from the same OTSCDS as in Figure 1 but using a simple linear interpolation.[7] The table shows the CDS spreads for the observed maturities and interpolated values. For the interval (0,6m], I could presume either a flat term structure or the same slope as that in the interval [6m,1y]. For this and the other cases of linear interpolation, I adopt the latter option.[8] I also assume a constant risk-free rate of 2% and recovery rate of 40%. Table 1 summarizes the final estimates of the CRDFs for the selected maturities and Figure 5 shows the results for all possible maturities. Notably, although different interpolation schemes (i.e., linear, PCHIP, and cubic spline) can be considered in the first step, the posterior bootstrapping process is always the same.

<Table 1 about here>

<Figure 5 about here>

[6] The last expression indicates that $A(T)$ tends to the present value of a risk-free daily annuity of $\$\Delta$ paid until time T. This result is a direct implication of Equation (4) and the remarks on $B(T)$ and $C(T)$.
[7] See www.santiagoforte.com for the Excel file containing this example.
[8] Please refer to Section 7.1 for a further discussion of the estimation criterion for the short end of the curve.

5. A Restricted Case: The Piecewise Constant Default Probability Model

This section describes the PWCDP model as a restricted version of the nonparametric model introduced above. First, I show that Equation System (9) produces the same results as does a model based on the CTSCDS that estimates the risk-neutral default probability at any time T, conditional on no previous default. If we denote $q(T)$ as an element of this term structure of forward risk-neutral default probabilities, then the risk-neutral survival probability at time T is

$$S(T) = \prod_{u=0}^{T/\Delta}[1 - q(u\Delta)], \tag{10}$$

whereas the risk-neutral default probability at time T (and not before) is

$$H(T) = q(T) \prod_{u=0}^{(T-\Delta)/\Delta} [1 - q(u\Delta)]. \tag{11}$$

Note that $q(0) = 0$. From Equations (10) and (11), and assuming that the risk-free interest process and default time are risk-neutrally independent, we obtain the following expressions for $A(T)$, $B(T)$, $C(T)$, and $E(T)$:

$$A(T) = \Delta \sum_{h=1}^{T/\Delta}\{Z(h\Delta)S[(h-1)\Delta]\} = \Delta \sum_{h=1}^{T/\Delta}\left\{Z(h\Delta)\prod_{u=0}^{h-1}[1 - q(u\Delta)]\right\}; \tag{12}$$

$$B(T) = \sum_{h=1}^{T/\Delta}\{Z(h\Delta)H(h\Delta)\} = \sum_{h=1}^{T/\Delta}\left\{Z(h\Delta)q(h\Delta)\prod_{u=0}^{h-1}[1 - q(u\Delta)]\right\}; \tag{13}$$

$$C(T) = Z(T)S(T) = Z(T)\prod_{u=0}^{T/\Delta}[1 - q(u\Delta)]\,; \tag{14}$$

$$E(T) = Z(T)S(T-\Delta) = Z(T) \prod_{u=0}^{(T-\Delta)/\Delta} [1 - q(u\Delta)]\,. \tag{15}$$

It is a relatively simple task (addressed in the Appendix) to show that Equations (12)–(15) satisfy both NAC1 and NAC2. Additionally, based on Equations (7), (12), and (13),

$$cds(T) = \frac{(1-\theta)\sum_{h=1}^{T/\Delta}\{Z(h\Delta)q(h\Delta)\prod_{u=0}^{h-1}[1-q(u\Delta)]\}}{\Delta\sum_{h=1}^{T/\Delta}\{Z(h\Delta)\prod_{u=0}^{h-1}[1-q(u\Delta)]\}}. \tag{16}$$

Equation (16) provides the break-even CDS spread for a contract with maturity T as a function of all forward risk-neutral default probabilities from 0 to T. From this equation, we can isolate $q(T)$ as a function of all previous probabilities:

$$q(T) = \frac{cds(T)\Delta\sum_{h=1}^{T/\Delta}\{Z(h\Delta)\prod_{u=0}^{h-1}[1-q(u\Delta)]\} - (1-\theta)\sum_{h=1}^{(T-\Delta)/\Delta}\{Z(h\Delta)q(h\Delta)\prod_{u=0}^{h-1}[1-q(u\Delta)]\}}{(1-\theta)Z(T)\prod_{u=0}^{(T-\Delta)/\Delta}[1-q(u\Delta)]}. \tag{17}$$

Hence, it is possible to bootstrap a full term structure of the $q(T)$ values from the CTSCDS using Equation (17). This term structure can be used to determine the core CRDFs from Equations (12)–(14) and price different single-name credit-risky securities, as described in Section 6. However, this task is both arduous and unnecessary because Equation System (9) more easily produces the same result. Furthermore, even if the intention is to estimate the term structure of the $q(T)$ and/or $S(T)$ values, it proves easier and faster to simply introduce two additional equations into Equation System (9):

$$S(T) = \frac{C(T)}{Z(T)}; \tag{18}$$

$$q(T) = 1 - \frac{S(T)}{S(T-\Delta)}. \tag{19}$$

Note that $S(0) = 1$.

Next, consider the implementation of a conventional PWCDP model. In our particular setting, this entails the following. First, it is assumed that $q(T)$ is piecewise constant in effect, where changes coincide with the maturity of the observable CDS spreads. Second, the term structure of the $q(T)$ values is estimated sequentially such that Equation (16) fits the observed CDS spreads perfectly. A bootstrapping process of this sort implies a sequence of root-search algorithms (one for each observed quote). Finally, the term structure of the core CRDFs and prices of different single-name credit-risky securities are determined based on the term structure of the $q(T)$ values obtained, as previously described.[9]

Comparing the two pricing approaches, the PWCDP model starts by imposing a piecewise constant profile on the term structure of $q(T)$, which ultimately leads to an effective interpolation of the observed CDS spreads (see Figure 1A). By contrast, the starting point of the nonparametric model is a particular interpolation scheme for these observed quotes, which finally implies a full term structure of the $q(T)$ values. Figure 6 shows a numerical example in which I estimate the term structures of $q(T)$ and $S(T)$ by assuming the same OTSCDS as in Figure 1 and the four pricing models: PWCDP, NP/Linear, NP/PCHIP, and NP/Spline. The PWCDP and NP/Linear models entail the most discontinuous term structures of the $q(T)$ values, whereas the NP/Spline model generates the smoothest term structure. Nevertheless, given the small marginal effect of $q(T)$ on $S(T)$, the term structure of $S(T)$ exhibits no evident jumps for any model.

The ability to generate a smooth term structure of forward risk-neutral default probabilities is a desirable property of credit risk pricing models. However, this could be considered to be of secondary importance if the main goals are simplicity and pricing accuracy. In this regard, I should emphasize that the nonparametric model is less restrictive, easier, and faster to implement than the PWCDP model. Moreover, as the empirical analysis in Section 7 demonstrates, it leads to a lower MPAPE. However, Section 6 first reviews the most evident applications of the term structure of CRDFs provided by these models.

<Figure 6 about here>

[9] The limit case in which $q(T)$ is a constant parameter equal to q can also be analyzed. In this instance, and based again on Equation (16), the CDS spread is also constant and given by $cds(T) = (1 - \theta)q/\Delta$. This result is nothing but the discrete-time version of the so-called *credit risk triangle* between the CDS spread, recovery rate, and constant hazard rate in a continuous-time model (e.g., O'Kane 2008). As is evident, this naïve version of a fully parametric model will only be consistent with a flat OTSCDS.

6. Applications

6.1. Pricing Credit Default Swap Contracts

The clearest application of the term structure of CRDFs is marking-to-market any position in a CDS contract. For a long position with a previously settled spread cds, this value is simply

$$V(T) = (1-\theta)B(T) - cdsA(T) = [cds(T) - cds]A(T). \tag{20}$$

This also implies a simple approach to estimate CDS returns (Berndt and Obreja 2010; Augustin et al. 2020; Lee et al. 2021).

6.2. Pricing Risky Bonds

Consider a risky bond with coupon b, nominal p, and maturity T. I denote T_m as the maturity of the m^{th} coupon payment, where $m = 1, \dots, M$, and $T_M = T$. The present value of this bond is

$$d(T) = b\sum_{m=1}^{M} C(T_m) + pC(T) + \theta pB(T). \tag{21}$$

The first term on the right-hand side of the equation reflects the present value of the stream of coupon payments. The second term accounts for the payment of the nominal amount at maturity in the case of no default. Finally, the last term incorporates the present value of the fractional recovery of the nominal value in the case of default.

6.3. Pricing Forward Credit Default Swap Contracts

Now, consider a forward CDS contract signed at current time 0 for credit protection between T_j and T_k, with $0 \leq T_j < T_k$. More precisely, the initiation date is T_j, conditional on $\tau^d > T_j$, so the first effective date with the accrual of premium payments and delivery of the bond in exchange for the bond's face value in the case of default is $T_j + \Delta$. The daily structure of this contract is, in fact, the structure described in Figures 3 and 4 for a spot contract. The sole difference is that the starting date is now T_j rather than 0, and the ending date is T_k. To derive the present value of the premium leg of the forward contract based on the CRDFs, I define (for any T^* and T, with $0 \leq T^* < T$):

- $A(T^*, T)$: The present value of the same asset class A paying a constant annuity of \$$\Delta$ every Δ years, but this time between T^* and T with the following conditions: (i) the first payment is at $T^* + \Delta$, conditional on $\tau^d > T^*$ (otherwise, the asset is liquidated at τ^d), and; (ii) provided that $\tau^d > T^*$, the last payment is at L_d^T (included).

Based on the definition of $A(T)$ and $A(T^*, T)$, it holds that

$$A(T^*, T) = A(T) - A(T^*). \quad (22)$$

If we use $fcds$ to denote the spread of the forward CDS contract described above, the present value of the premium leg is:

$$X(T_j, T_k) = fcdsA(T_j, T_k). \quad (23)$$

We can also derive the present value of the protection leg based on the CRDFs. I define:

- $B(T^*, T)$: The present value of the same asset class B paying \$1 at τ^d, provided this time that $T^* < \tau^d \leq T$.

From the definition of $B(T)$ and $B(T^*, T)$, it must hold that

$$B(T^*, T) = B(T) - B(T^*), \quad (24)$$

and the present value of the protection leg is:

$$Y(T_j, T_k) = (1 - \theta)B(T_j, T_k). \quad (25)$$

The value of a long position in the forward CDS contract is thus:

$$FV(T_j, T_k) = (1 - \theta)B(T_j, T_k) - fcdsA(T_j, T_k). \quad (26)$$

By imposing $FV(T_j, T_k) = 0$, we finally obtain the break-even forward CDS spread:

$$fcds(T_j, T_k) = \frac{(1 - \theta)B(T_j, T_k)}{A(T_j, T_k)}. \quad (27)$$

It is worth noting that $fcds(0, T) = cds(T)$.

6.4. A Note on Portfolio Management

The results on the pricing of single-name credit-risky securities above apply regardless of the pricing model used to determine the CRDFs. However, the nonparametric model I introduce here offers clear advantages for portfolio management. The combination of a CTSCDS (obtained directly from the OTSCDS) and the term structure of risk-free interest rates provides a direct estimate of the term structure of CRDFs. Moreover, as the prices of these securities are simple functions of the CRDFs, the model allows for a straight mapping between the observable market risk factors (OTSCDS and term structure of risk-free interest rates) and prices of the most common single-name credit-risky securities (spot and forward CDS contracts and risky bonds). The final implication is the possibility of translating the predicted distribution function for such market risk factors into a distribution function for the values of different credit-risky portfolios using Monte Carlo simulations.[10] By extension, this also represents an easy path for integrating market and credit risk.

7. Semiparametric vs. Nonparametric Estimation: Relative Pricing Errors

7.1. Data and Comparison Criterion

This section compares the pricing performance of the conventional PWCDP model with that of my proposed nonparametric model. In the latter case, I consider different interpolation schemes for the ex-ante estimation of the CTSCDS. The initial sample consists of the 107 companies included in the CDX (NA.IG and/or NA.HY) from 2010 to 2019. I collect USD-denominated CDS spreads with an MR/MR14 clause daily for all possible maturities: 6m, 1y, 2y, 3y, 4y, 5y, 7y, 10y, 15y, 20y, and 30y. In principle, this implies 2,569 observations of such CDS spreads for each company; in practice, the CDS spreads for all maturities are not always available. To avoid dealing with missing data, I restrict the analysis to company-dates with all quotes available. I also remove three companies because of convergence problems in the implementation of the PWCDP model on several dates. Thus, the final sample contains 248,218 company-date records of the OTSCDS without missing quotes or PWCDP model convergence problems, representing 90.3% of the initial 274,883 records. Table 2 presents the descriptive statistics of the CDS spreads. I compute the forward rates from the Treasury zero-coupon yield curve provided by the Federal Reserve Board at a daily frequency.

<Table 2 about here>

[10] Clearly, this extension to portfolio management should incorporate the probability of a default state at the future pricing date and, thus, account for the connection between historical/current CDS levels and the probability of future default.

I evaluate the accuracy of the four pricing models—PWCDP, NP/Linear, NP/PCHIP, and NP/Spline—based on their relative pricing errors. Unlike the illustrative example in Section 1, the true CTSCDS is unknown. Because only a limited number of observed quotes are available, the comparison proceeds as follows. For each company-date observation, I estimate the four models using all but the 6m-CDS spread. Next, I use the results to determine the 6m-CDS spread predicted by each model, with the pricing error gauged through a comparison with the actual quote. I repeat this process for other available maturities to obtain the final sample of pricing errors.

Three important clarifications about this empirical study need to be made. First, while it compares a pricing model (PWCDP) and three direct interpolation schemes for the observed quotes (linear, PCHIP, and cubic spline), because the nonparametric model can reproduce any of these CTSCDS, the analysis represents, in effect, a comparison between the conventional PWCDP model and three versions of the nonparametric model. Second, although the overall results allow the ranking of the models according to their pricing accuracy, the values obtained can be expected to underestimate their real precision across the credit curve because some information that can be used to estimate them is always ignored. In addition, these *hypothetical* errors are typically estimated at or close to the midpoint of two observed maturities, where the results of the illustrative example in Figure 1 suggest that pricing errors tend to peak. Third, while I consider pricing errors for the 6m maturity (the first OTSCDS element) for comparison purposes, I exclude those for the 30y maturity (the final element). This is because, in practice, the lowest possible CDS contract maturity that might need to be priced is one day. Hence, whatever the lowest available maturity within the OTSCDS, some form of *extrapolation* is always required to complete the short end of the CTSCDS. By contrast, the maturity of an existing CDS contract will never be higher than that of the last available quote in the OTSCDS. Therefore, potential pricing errors beyond the longest available maturity do not need to be investigated, which may distort the conclusions. However, in some cases, the proposed extrapolation to the left of the curve may lead to negative values. I take several steps to prevent this problem in the estimation of the 6m-CDS spread. A positive value for the resulting one-day spread validates the estimated 6m-spread. Otherwise, I assume that the CDS spread for a contract with zero maturity is zero, and this hypothetical quote is incorporated as part of the ordinary interpolation process. Notably, for the linear and PCHIP interpolation schemes, this rule guarantees a positive value for the estimated 6m-CDS spread because both satisfy the shape-preserving property. However, negative values are still possible when using the cubic spline method.

7.2. Results

Table 3 provides the main descriptive statistics of the PAPEs. Among the four pricing approaches, the NP/PCHIP model has the lowest MPAPE, whereas the PWCDP model has the highest MPAPE. Notably, even the simplest, linear version of the nonparametric approach offers more accurate results on average than the conventional PWCDP model.

<Table 3 about here>

Figure 7 presents an in-depth analysis of the pricing errors generated by each model. Figure 7A plots the MPAPE as a function of contract maturity. I find that the PWCDP model underperforms in all versions of the nonparametric approach for the shortest maturities. However, these differences tend to decline and even revert for the longest maturities. Specifically, the nonparametric models (including NP/Linear) outperform the PWCDP model in the interval (0,5y]. For the remaining terms, the NP/PCHIP and PWCDP models offer similar results, which, in turn, outperform the NP/Linear and NP/Spline models. The accuracy of any of the models in the range (0,5y] is particularly relevant. As 5y is by far the most traded maturity, most existing CDS contracts have a remaining maturity within that interval. As a complementary analysis, Figure 7B depicts the MPAPE by credit risk level proxied by the 5y-CDS spread. The results confirm that regardless of the credit risk level, the PWCDP model is the less accurate approach on average, while the NP/PCHIP model is the most accurate.

Based on the empirical evidence on bid-ask spreads in Du et al. (2019), I conclude that the differences in the pricing errors are not economically significant. While such information is not available for my sample, they analyze a similar sample and report relative bid-ask spreads for the 1y, 2y, 3y, 7y, and 10y maturities of 0.7235, 0.5285, 0.2652, 0.1321, and 0.1266, respectively (though do not report information for other maturities; see their Internet Appendix). As these values are already higher than the (upward-biased) estimates of the MPAPE reported in Figure 7A for all the models, it can certainly be inferred that pricing error differences do exist but are economically insignificant. Thus, the main conclusions of this empirical analysis are as follows. Relative to the conventional PWCDP model, the nonparametric approach represents a much more flexible, simpler, and faster (as well as reliable) pricing method. However, these benefits do not come at the cost of lower precision. Indeed, the nonparametric approach, particularly the NP/PCHIP model, also has the advantage of an effective (although not economically significant) reduction in pricing errors.

<Figure 7 about here>

8. Conclusions

This study introduces a simple nonparametric approach to pricing CDS contracts and other single-name credit-risky securities. By this means, it contributes to the extant literature in which pricing models are either parametric or semiparametric. Similar to the traditional estimation of implied discount factors in risk-free bond prices, this method provides direct estimates of CRDFs from a prespecified CTSCDS. Its implementation is based exclusively on closed-form solutions, removing the need for root-search algorithms or other forms of optimization. Empirical evidence from a large sample of companies over 2010–2019 confirms that the new method produces fewer pricing errors than a conventional PWCDP model, which can be seen as a restricted and computationally demanding version of the nonparametric approach presented here. However, the reduction in pricing errors is not economically significant and therefore is only an additional benefit of the proposed method. The main advantages of the nonparametric approach are its flexibility, simplicity, and estimation speed.

The present study has some limitations, leaving the door open for future research. For example, while the proposed method is an effective alternative to semiparametric models for marking-to-market CDS positions, I do not provide a comparison with parametric models. However, I also do not present the nonparametric approach as a substitute for parametric models that have their own paths beyond pricing. Additionally, the new method accommodates any prespecified CTSCDS and I explore only the performance of certain common interpolation schemes. In reality, this method allows the user to determine the most convenient technique for the ex-ante estimation of the CTSCDS. The election depends on the problem, and the optimal choice may not be among the non-exhaustive list of possibilities I consider here. If the intention is marking-to-market CDS contracts with up to 5y maturities, simple linear interpolation seems sufficiently accurate (or at least more accurate than the standard PWCDP model) and straightforward to implement. However, this implies a discontinuous term structure for the forward risk-neutral default probability. Spline interpolation generates a smooth term structure for such probabilities; however, it is less accurate. One interesting possibility I do not consider here is the estimation of the CTSCDS using a Nelson–Siegel or Svensson-like model. Compared with a straight interpolation between the observed CDS spreads, it has the disadvantage of involving a parametric model, an optimization process, and pricing errors in the observed CDS spreads. However, a parametric model of this type could be useful for certain applications and a sensible compromise between simplicity, smoothness, and accuracy.

Appendix

This appendix demonstrates that Equations (12)–(15) satisfy both NAC1 and NAC2. Regarding NAC1:

$$A(T) = \Delta \sum_{h=1}^{T/\Delta} \{Z(h\Delta)S[(h-1)\Delta]\}$$

$$= \Delta \sum_{h=1}^{(T-\Delta)/\Delta} \{Z(h\Delta)S[(h-1)\Delta]\} + \Delta Z(T)S(T-\Delta)$$

$$= A(T-\Delta) + \Delta E(T).$$

Regarding NAC2:

$$C(T) + B(T) - B(T-\Delta) = Z(T)S(T) + \sum_{h=1}^{T/\Delta} \{Z(h\Delta)H(h\Delta)\} - \sum_{h=1}^{(T-\Delta)/\Delta} \{Z(h\Delta)H(h\Delta)\}$$

$$= Z(T)S(T) + Z(T)H(T)$$

$$= Z(T)\left\{\prod_{u=0}^{T/\Delta}[1-q(u\Delta)] + q(T)\prod_{u=0}^{(T-\Delta)/\Delta}[1-q(u\Delta)]\right\}$$

$$= Z(T)\prod_{u=0}^{(T-\Delta)/\Delta}[1-q(u\Delta)]$$

$$= Z(T)S(T-\Delta)$$

$$= E(T).$$

References

Augustin, P., Saleh, F., and Haohua, X. (2020). CDS returns. *Journal of Economic Dynamic and Control, 118*, Article 103977.

Berndt, A., and Obreja, I. (2010). Decomposing European CDS returns. *Review of Finance, 14(2)*, 189–233.

Chen, R.R., Cheng, X., and Wu, L. (2013). Dynamic interactions between interest-rate and credit risk: theory and evidence on the credit default swap term structure. *Review of Finance, 17(1)*, 403–441.

Du, D., Elkamhi, R., and Ericsson, J. (2019). Time-varying asset volatility and the credit spread puzzle. *Journal of Finance, 74(4)*, 1841–1885.

Duffie, D. (1999). Credit swap valuation. *Financial Analysts Journal, 55(1)*, 73–87.

Hull J., and White, A. (2003). The valuation of credit default swap options. *Journal of Derivatives, 10(3)*, 40–50.

Jarrow, R.A., Lando, D., and Turnbull, S.M. (1997). A Markov model for the term structure of credit risk spreads. *Review of Financial Studies, 10(2)*, 481–523.

Jarrow, R.A., Li, H., Ye, H., and Hu, M. (2019). Exploring mispricing in the term structure of CDS spreads. *Review of Finance, 23(1)*, 161–198.

Jarrow, R.A., and Turnbull, S.M. (1995). Pricing derivatives on financial securities subject to credit risk. *Journal of Finance, 50(1)*, 53–85.

Lando, D. (1998). On Cox processes and credit risky securities. *Review of Derivatives Research, 2*, 99–120.

Lee, J., Naranjo, A., and Sirmans, S. (2021). CDS momentum: slow-moving credit ratings and cross-market spillovers. *Review of Asset Pricing Studies, 11(2)*, 352–401.

Merton R.C. (1974). On the pricing of corporate debt: the risk structure of interest rates. *Journal of Finance, 29(2)*, 449–470.

O'Kane D. (2008). *Modelling single-name and multi-name credit derivatives*. John Wiley & Sons Ltd: Chichester, England.

O'Kane D., and Turnbull, S. (2003). Valuation of credit default swaps. Lehman Brothers.

Pan, J., and Singleton, K.H. (2008). Default and recovery implicit in the term structure of sovereign CDS spreads. *Journal of Finance, 63(5)*, 2345–2384.

Tables

Table 1. Numerical example of bootstrapping the CRDFs.

Obs. Mat.	T	$cds(T)$	$A(T)$	$B(T)$	$C(T)$
	0	-	**0.000000**	**0.000000**	**1.000000**
	1/365	4.25	0.002740	0.000002	0.999943
	2/365	4.27	0.005479	0.000004	0.999887
	…	…	…	…	…
	182/365	9.16	0.495985	0.000757	0.989323
6m	**183/365**	**9.19**	0.498695	0.000764	0.989262
	184/365	9.21	0.501405	0.000770	0.989202
	…	…	…	…	…
	364/365	14.10	0.986473	0.002318	0.977952
1y	**1**	**14.13**	0.989152	0.002329	0.977887
	…	…	…	…	…
2y	**2**	**28.51**	1.954228	0.009284	0.951630
	…	…	…	…	…
3y	**3**	**44.85**	2.890680	0.021609	0.920575
	…	…	…	…	…
4y	**4**	**60.54**	3.794190	0.038286	0.885828
	…	…	…	…	…
5y	**5**	**74.44**	4.661809	0.057834	0.848927
	…	…	…	…	…
7y	**7**	**95.82**	6.285719	0.100387	0.773895
	…	…	…	…	…
10y	**10**	**114.70**	8.451172	0.161557	0.669415
	…	…	…	…	…
15y	**15**	**127.46**	11.438759	0.243001	0.528218
	…	…	…	…	…
20y	**20**	**131.79**	13.803296	0.303180	0.420746
	…	…	…	…	…
30y	**30**	**134.81**	17.203669	0.386542	0.269375

This table presents a subsample of the numerical example results, where the term structures of $A(T)$, $B(T)$, and $C(T)$ are estimated based on the CTSCDS and Equation System (9). In this case, the CTSCDS is obtained through a linear interpolation of CDS spreads with observed maturities (Obs. Mat.). Observed spreads and initial CRDF values are indicated in bold. The example assumes a constant risk-free rate of 2% and recovery rate of 40%.

Table 2. Main descriptive statistics: CDS spreads.

	cds(0.5)	cds(1)	cds(2)	cds(3)	cds(4)	cds(5)	cds(7)	cds(10)	cds(15)	cds(20)	cds(30)
Mean	36.01	45.34	70.10	97.56	125.13	152.17	181.61	196.81	204.45	208.19	210.08
Median	10.80	14.19	25.79	40.24	56.66	74.80	100.78	117.49	126.53	132.04	135.48
Min	0.82	1.06	3.04	4.53	6.55	9.84	20.52	28.18	28.89	29.63	32.26
Max	19,942.33	17,395.30	14,307.61	12,710.33	11,810.71	11,341.91	10,772.98	10,277.27	9,777.48	9,510.61	9,297.33
SD	162.70	166.18	183.37	200.47	215.00	227.81	228.56	222.12	216.04	212.30	209.61

This table presents the main descriptive statistics of the CDS spreads in the sample: the mean, median, minimum, maximum, and standard deviation.

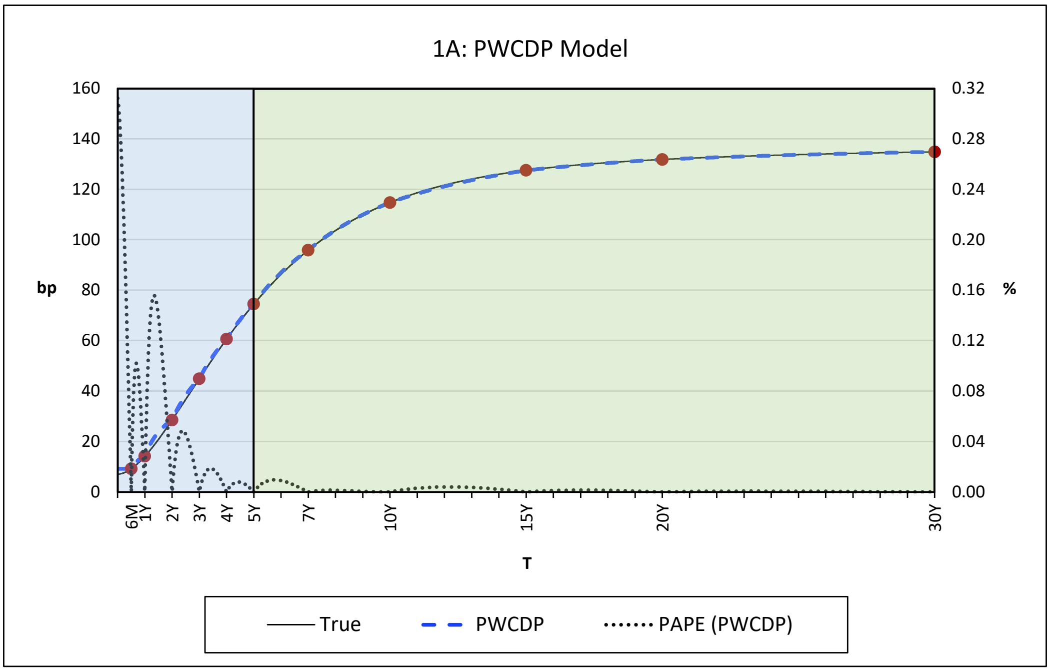
Table 3. Main descriptive statistics: PAPEs by pricing model.

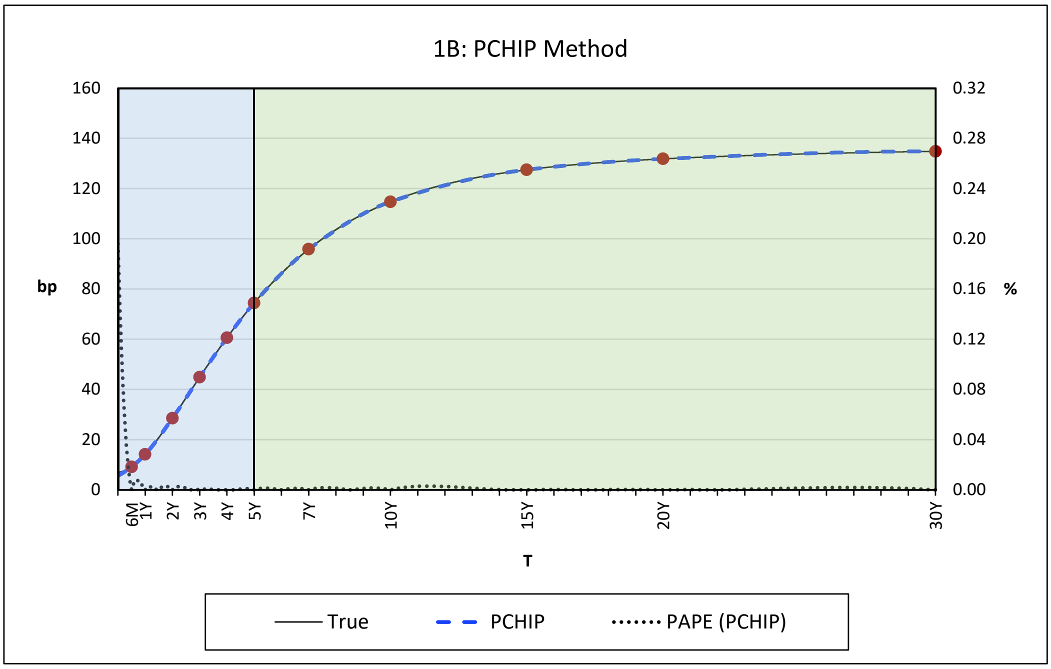
	PWCDP	NP/Linear	NP/PCHIP	NP/Spline
Mean	0.16	0.08	0.06	0.10
Median	0.07	0.04	0.03	0.03
Min	0.00	0.00	0.00	0.00
Max	18.71	17.05	17.45	30.82
SD	0.26	0.18	0.18	0.33

This table presents the main descriptive statistics of the PAPEs by pricing model: PWCDP, NP/Linear, NP/PCHIP, and NP/Spline. The PAPE for each observed maturity is estimated by ignoring a specific quote in the estimation process, and the actual and predicted CDS spreads for that maturity are compared. The reported statistics correspond to the pricing errors for 6m, 1y, 2y, 3y, 4y, 5y, 7y, 10y, 15y, and 20y maturities.

Figures

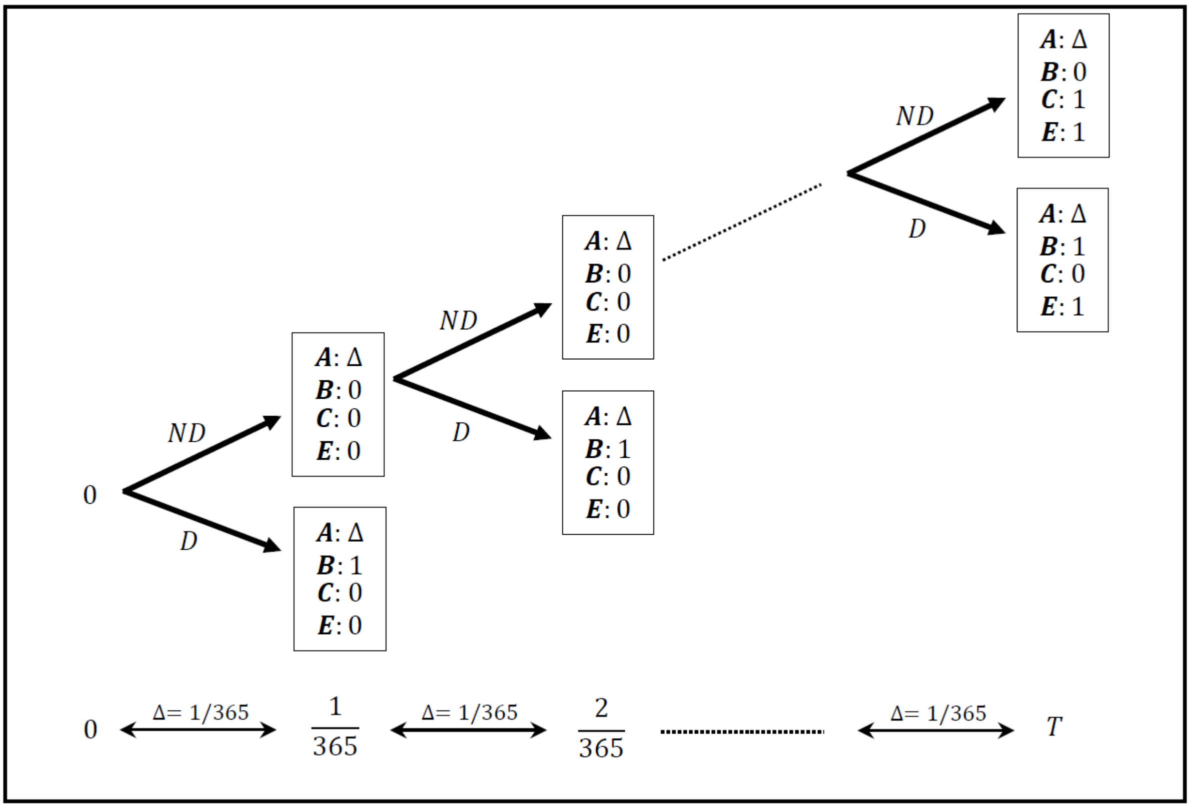
Figure 1. Example of estimation approaches for the complete term structure of CDS spreads.





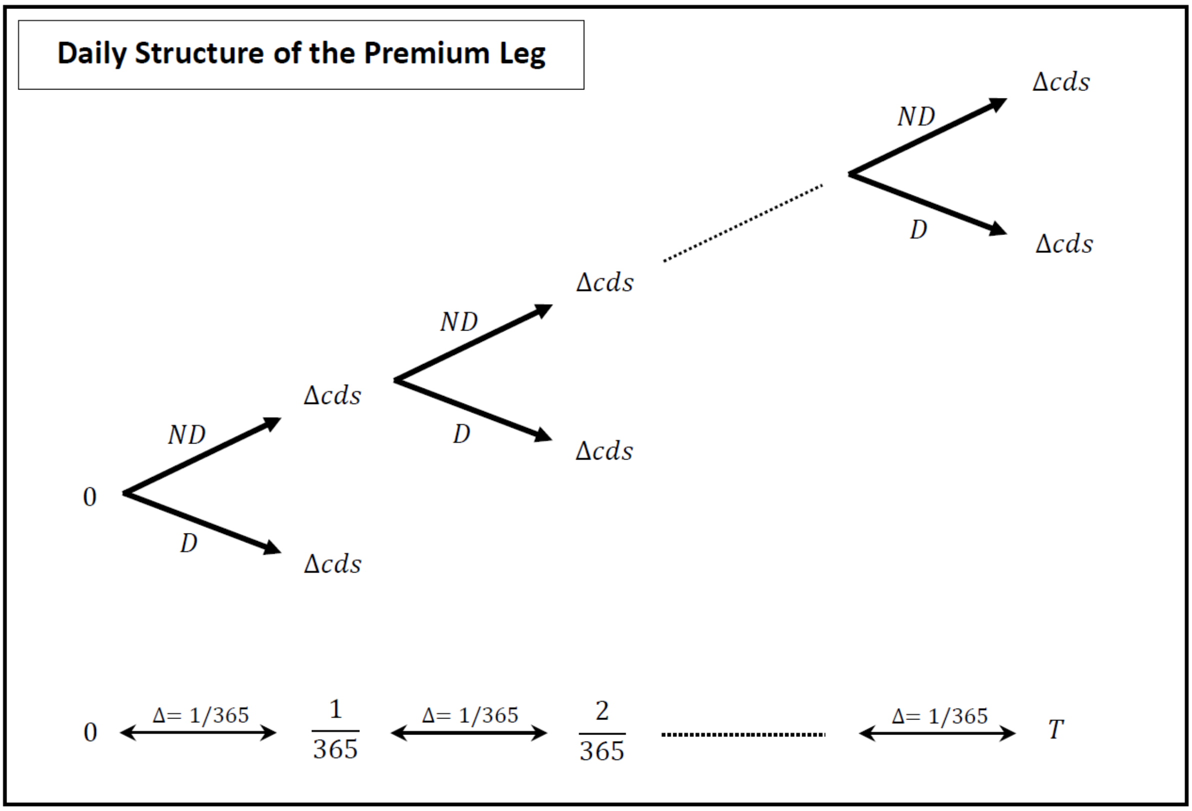
This figure provides an example of two possible estimation approaches for the CTSCDS (True, black solid line, left axis) based on the OTSCDS (red points, left axis): the PWCDP model (Figure 1A; blue dashed line, left axis) and PCHIP method (Figure 1B; blue dashed line, left axis). It is assumed that the actual CTSCDS corresponds to a particular parametrization of the Svensson model ($\beta_0 = 140$; $\beta_1 = -133$; $\beta_2 = -325$; $\beta_3 = 275$; $\alpha_1 = 2.2$; $\alpha_2 = 3.1$). This figure also shows the PAPEs for each estimation method (black dotted line, right axis).

Figure 2. Payment structures of assets A, B, C, and E with maturity T.



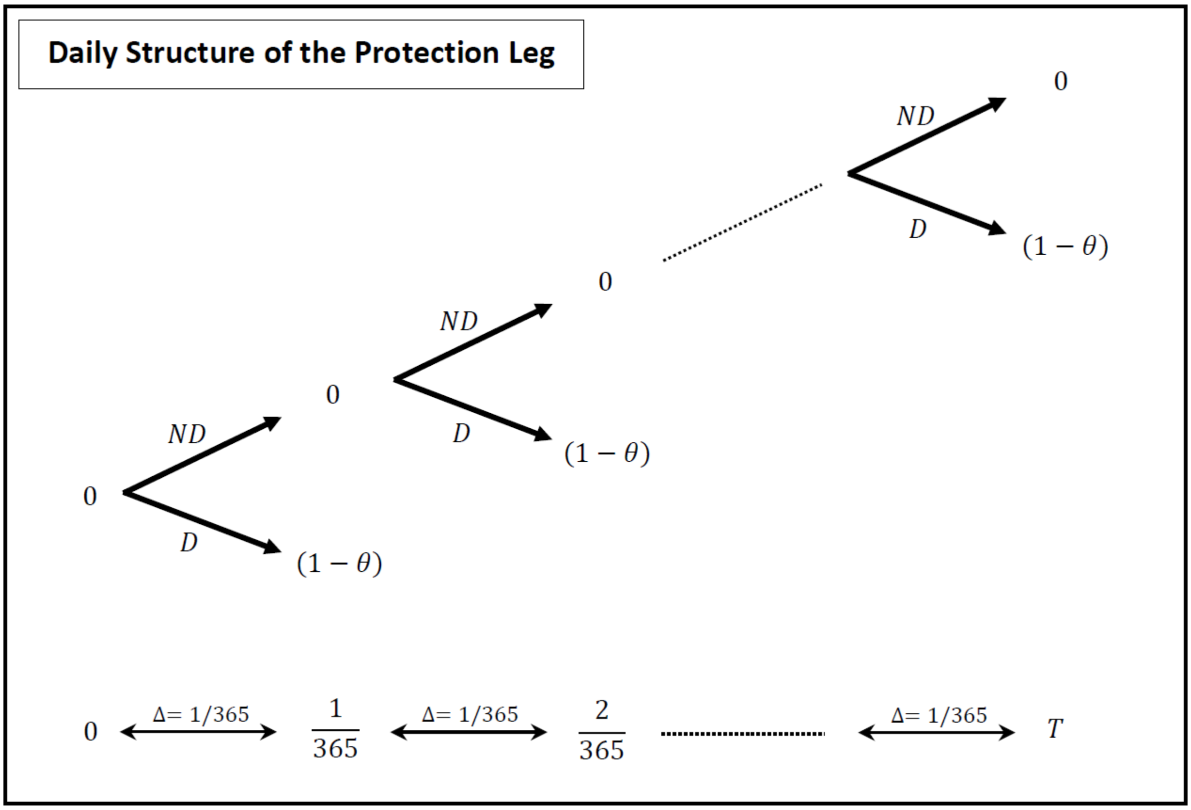
This figure presents the payment structures of assets A, B, C, and E with maturity $T > 0$. The possible outcomes for each day are no default (ND) and default (D).

Figure 3. Daily structure of the premium leg in a CDS contract with maturity T.



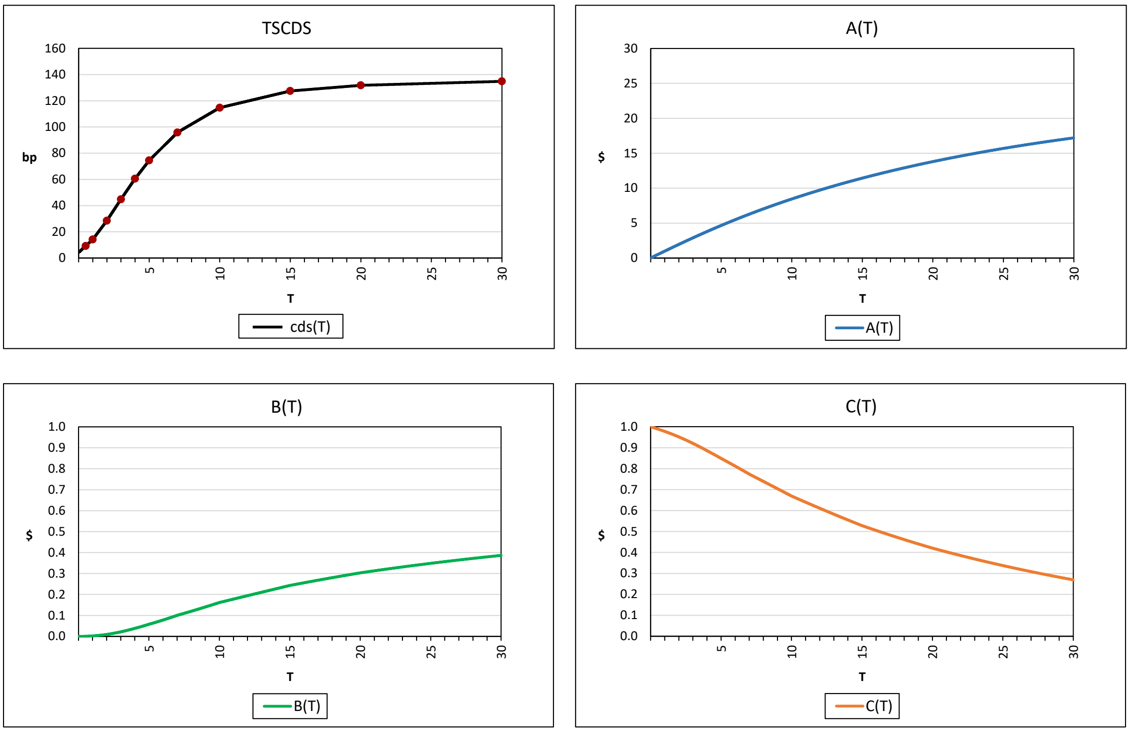
This figure shows the daily structure of the premium leg in a CDS contract with maturity $T > 0$. The possible outcomes for each day are no default (ND) and default (D).

Figure 4. Daily structure of the protection leg in a CDS contract with maturity T.



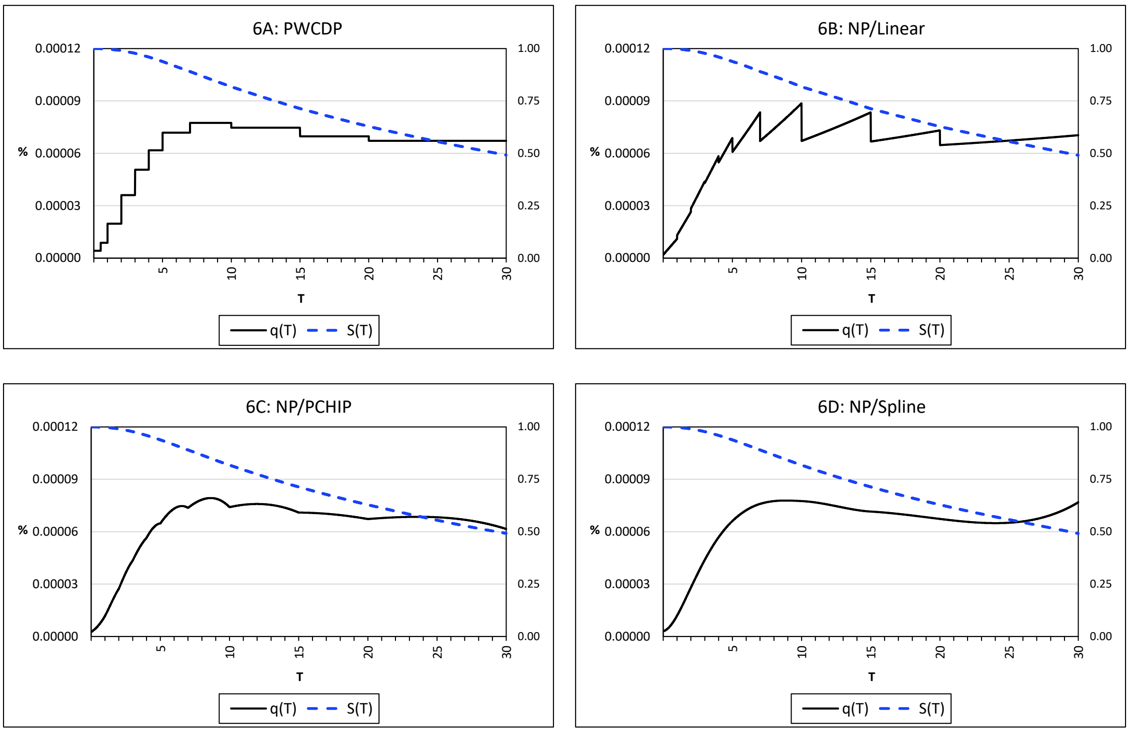
This figure shows the daily structure of the protection leg in a CDS contract with maturity $T > 0$. The possible outcomes for each day are no default (ND) and default (D).

Figure 5. Numerical example of bootstrapping the CRDFs.



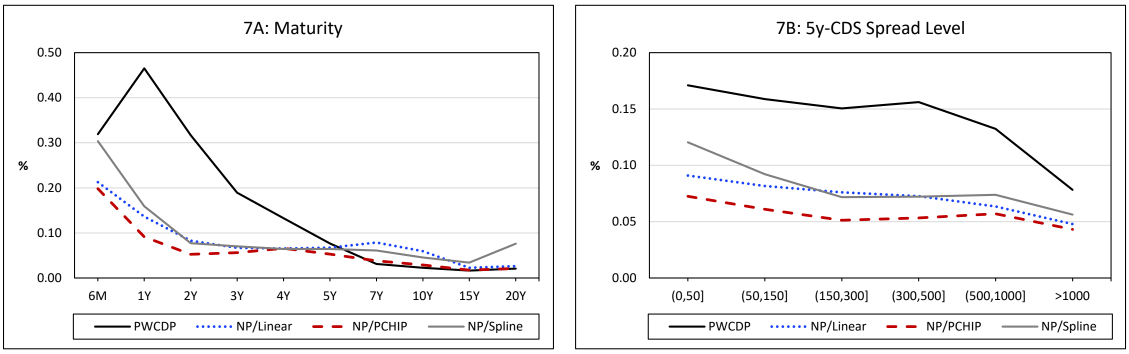
This figure plots the numerical example results, where the term structures of $A(T)$, $B(T)$, and $C(T)$ are estimated based on the CTSCDS and Equation System (9). Red points indicate that the CDS spread corresponds to an observed maturity. In this case, the CTSCDS was obtained via a linear interpolation of the observed quotes. The example assumes a constant risk-free rate of 2% and recovery rate of 40%.

Figure 6. Example of the term structures of forward risk-neutral default and survival probabilities by pricing model.



This figure plots the term structures of $q(T)$ (black solid line, left axis) and $S(T)$ (blue dashed line, right axis) for the four pricing models: PWCDP (Figure 6A), NP/Linear (6B), NP/PCHIP (6C), and NP/Spline (6D). All cases were assumed to have the same OTSCDS.

Figure 7. MPAPE by maturity and 5y-CDS spread level.



This figure plots the MPAPE by maturity (Figure 7A) and 5y-CDS spread level (Figure 7B). The pricing models considered were PWCDP (black solid line), NP/Linear (blue dotted line), NP/PCHIP (red dashed line), and NP/Spline (gray solid line).